Wake field of an electron bunch moving parallel to a dielectric cylinder

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The wake field of an electron bunch moving parallel to the axis of a dielectric cylinder is being considered. It is shown that for a relativistic bunch ($\gamma \gg 1$) the circular harmonic of order zero contributes a decelerating force inversely proportional to γ, whereas the circular harmonics of nonzero order contribute a γ-independent force. Moreover, the wake linked to the circular harmonic of order zero may grow in space in case the dielectric cylinder consists of an active medium; however, this growth rate does not depend on the value of γ. On the other hand, no growth is anticipated for the case of circular harmonics of nonzero order.

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I. INTRODUCTION

Acceleration of electrons by radiation at optical wavelengths is one of the most promising alternatives for future electron acceleration. Generally speaking, optical schemes may be divided into two main groups: in the case of *plasma-based* schemes, a laser pulse is injected into a plasma where it excites a space-charge wake that, in turn, may accelerate a trailing bunch of electrons [1–4]. Another group corresponds to various *inverse radiation* processes such as inverse Cerenkov [5–7], inverse free-electron laser (IFEL) [8,9], and inverse transition radiation [10,11]. In the case of an inverse radiation process, the laser pulse is injected at identical conditions as when the radiation is emitted by electrons propagating within the structure. For example, in an IFEL, the laser pulse exhibits a polarization and a wavelength such that in the presence of the wiggler, the motion of the electron is synchronous with the wave, the phase corresponding to an accelerating force. In all these laser-driven systems, energy stored in an active medium is transformed into radiation inside the laser cavity being further used for acceleration in various structures.

It was suggested in Refs. [12,13] to directly use energy stored in an active medium in order to accelerate electrons. Specifically, it was demonstrated that a Cerenkov wake, generated by a small trigger bunch, may be amplified by the medium. A second bunch trailing behind may be accelerated by the amplified wake. The concept was demonstrated within the framework of a *linear* theory when the charged-particle moves in the active medium. A possible practical experiment is to launch a bunch of electrons parallel to a dielectric cylinder that may be active, e.g., a Nd:YAG rod, and examine the acceleration of electrons by the amplified wake. Since the transverse dimension of the bunch (100 μm diameter) is significantly smaller than that of the dielectric rod (6 mm), many nonsymmetric modes may be excited.

It is the purpose of this study to determine the radiation characteristics generated by a relativistic bunch of electrons moving parallel to a dielectric cylinder.

II. MODEL FORMULATION

Consider a cylinder of radius R consisting of dielectric material (ϵ). The axis of a cylindrical coordinate system (r,ϕ,z) coincides with that of the cylinder. Parallel to this axis, at a radius $r=h>R$ and at an angle $\phi=\phi_0$, a point charge is moving at a velocity V—see Fig. 1. In its motion, the point charge generates a current density

$$J_z(r,\phi,z;t) = -qV\frac{1}{r}\delta(r-h)\,\delta(\phi-\phi_0)\,\delta(z-Vt) \quad (1)$$

whose time-Fourier transform reads

$$J_z(r,\phi,z;\omega) = -q\frac{1}{r}\delta(r-h)\,\delta(\phi-\phi_0)\frac{1}{2\pi}\,e^{-j(\omega/V)z}. \quad (2)$$

In the *absence* of the cylinder this current density excites a primary (superscript p) magnetic vector potential $A_z^{(p)}(r,\phi,z;\omega)$ which is a solution of the equation

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] A_z^{(p)}(r,\phi,z;\omega) = -\mu_0 J_z(r,\phi,z;\omega). \quad (3)$$

In the cylindrical coordinate system resorted to, this solution reads

$$A_z^{(p)}(r,\phi,z;\omega) = -\frac{q\mu_0}{(2\pi)^2}\,e^{-j(\omega/V)z}\sum_{\nu=-\infty}^{\infty} e^{j\nu(\phi-\phi_0)} \times \begin{cases} I_\nu(\Gamma h)K_\nu(\Gamma r) & r>h, \\ K_\nu(\Gamma h)I_\nu(\Gamma r) & r<h, \end{cases} \quad (4)$$

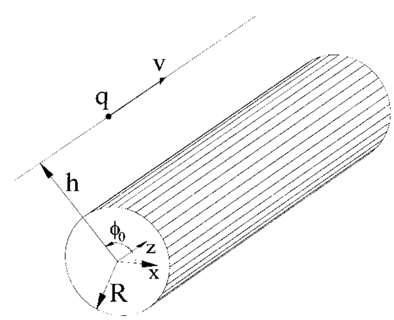

FIG. 1. Basic setup of the system under consideration; a dielectric cylinder (e.g., Nd:YAG) of radius R and a bunch of electrons injected parallel to the axis at a radius $r=h$ and an angle $\phi=\phi_0$.

wherein $\Gamma = |\omega|/c\gamma\beta$, $\beta = V/c$, $\gamma = [1-\beta^2]^{-1/2}$, $I_\nu(\xi)$, and $K_\nu(\xi)$ are modified Bessel functions of order ν, of the first and second type, respectively.

At the surface of the cylinder, the tangential components of the primary field are given by

$$E_z^{(p)}(r=R,\phi,z;\omega) = \sum_{\nu=-\infty}^{\infty} \left[\frac{j\omega}{\gamma^2\beta^2} I_{gn}(\Gamma R) \right] e^{j\nu(\phi-\phi_0)} e^{-j(\omega/V)z} a_\nu ,$$

$$E_\phi^{(p)}(r=R,\phi,z;\omega) = \sum_{\nu=-\infty}^{\infty} \left[\frac{-j\nu c/R}{\beta} I_\nu(\Gamma R) \right] e^{j\nu(gf-\phi_0)} e^{-j(\omega/V)z} a_\nu , \tag{5}$$

$$H_\phi^{(p)}(r=R,\phi,z;\omega) = \sum_{\nu=-\infty}^{\infty} \left[\frac{-1}{\mu_0} \Gamma \dot{I}_\nu(\Gamma R) \right] e^{j\nu(\phi-\phi_0)} e^{-j(\omega/V)z} a_\nu ,$$

where $a_\nu \equiv -(q\mu_0/4\pi^2) K_\nu(\Gamma h)$; $\dot{I}_\nu(\xi)$ stands for the derivative of $I_\nu(\xi)$ with respect to the argument (ξ).

The *presence* of the cylinder alters the field distribution in the whole space. This change is due to the so-called secondary (superscript s) field whose longitudinal components read

$$E_z^{(s)}(r,\phi,z;\omega) = e^{-j(\omega/V)z} \sum_{\nu=-\infty}^{\infty} e^{j\nu(\phi-\phi_0)} \times \begin{cases} A_\nu K_\nu(\Gamma r) & r>R, \\ B_\nu J_\nu(\Lambda r) & r<R, \end{cases} \tag{6}$$

$$H_z^{(s)}(r,\ \phi,\ z;\ \omega) = e^{-j(\omega/V)z} \sum_{\nu=-\infty}^{\infty} e^{j\nu(\phi-\phi_0)} \times \begin{cases} C_\nu K_\nu(\Gamma r) & r>R, \\ D_\nu J_\nu(\Lambda r) & r<R, \end{cases} \tag{7}$$

wherein $\Lambda = |\omega|\sqrt{\epsilon - 1/\beta^2}/c$; $J_\nu(\xi)$ is the Bessel function of the first kind and order ν. Formulation of the boundary conditions at $r=R$ requires—in addition to Eqs. (6) and (7)—also the determination of the azimuthal components of the electromagnetic field; these are given by

$$E_\phi^{(s)} = e^{-j(\omega/V)z} \sum_{\nu=-\infty}^{\infty} e^{j\nu(\phi-\phi_0)} \begin{cases} \dfrac{-\gamma^2\beta^2}{\omega^2/c^2} \left[j\omega\mu_0 \Gamma C_\nu \dot{K}_\nu(\Gamma r) + \dfrac{j\nu}{r} \left(\dfrac{-j\omega}{V} \right) A_\nu K_\nu(\Gamma r) \right] & r<R, \\ \dfrac{1}{\Lambda^2} \left[j\omega\mu_0 \Lambda D_\nu \dot{J}_\nu(\Lambda r) + \dfrac{j\nu}{r} \left(\dfrac{-j\omega}{V} \right) B_\nu J_\nu(\Lambda r) \right] & r<R, \end{cases} \tag{8}$$

$$H_\phi^{(s)} = e^{-j(\omega/V)z} \sum_{\nu=-\infty}^{\infty} e^{j\nu(\phi-\phi_0)} \begin{cases} \dfrac{-\gamma^2\beta^2}{\omega^2/c^2} \left[\dfrac{j\nu}{r} \left(\dfrac{-j\omega}{V} \right) C_\nu K_\nu(\Gamma r) - j\omega\epsilon_0 \Gamma A_\nu \dot{K}_\nu(\Gamma r) \right] & r>R, \\ \dfrac{1}{\Lambda^2} \left[\dfrac{j\nu}{r} \left(\dfrac{-j\omega}{V} \right) D_\nu J_\nu(\Lambda r) - j\omega\epsilon_0 \epsilon \Lambda B_\nu \dot{J}_\nu(\Lambda r) \right] & r<R. \end{cases} \tag{9}$$

where $\dot{J}_\nu(\xi)$ stands for the derivative with respect to the argument ξ of the Bessel function of the first kind and order ν; $\dot{K}_\nu(\xi)$ stands for the derivative of the modified Bessel function of the second kind, order ν, also with respect to ξ.

Continuity of the tangential components imposes four conditions required in order to determine the four unknown amplitudes A_ν, B_ν, C_ν, and D_ν. Being interested only in the longitudinal reaction force acting on the point charge, we shall state here only the explicit expression of the scattered amplitude of $E_z^{(s)}$, namely,

$$A_\nu = \frac{j\omega a_\nu}{\gamma^2\beta^2} \frac{\dfrac{\nu\gamma^2}{\Omega} \left(1 + \dfrac{1}{\bar{\epsilon}\gamma^2\beta^2} \right)^2 J_\nu^2 K_\nu I_\nu - \left[\dot{I}_\nu J_\nu + \dfrac{\epsilon}{\gamma\beta\sqrt{\bar{\epsilon}}} I_\nu \dot{J}_\nu \right] \left[\dot{K}_\nu J_\nu + \dfrac{1}{\gamma\beta\sqrt{\bar{\epsilon}}} K_\nu \dot{J}_\nu \right]}{\left[\dfrac{\nu\gamma}{\Omega} \left(1 + \dfrac{1}{\bar{\epsilon}\gamma^2\beta^2} \right) J_\nu K_\nu \right]^2 - \left[\dot{K}_\nu J_\nu + \dfrac{\epsilon}{\gamma\beta\sqrt{\bar{\epsilon}}} K_\nu \dot{J}_\nu \right] \left[\dot{K}_\nu J_\nu + \dfrac{1}{\gamma\beta\sqrt{\bar{\epsilon}}} K_\nu \dot{J}_\nu \right]}, \tag{10}$$

wherein $\Omega \equiv \omega R/c$, $J_\nu \equiv J_\nu(\Omega\sqrt{\epsilon})$, $I_\nu \equiv I_\nu(\Omega/\gamma\beta)$, and $K_\nu \equiv K_\nu(\Omega/\gamma\beta)$.

Along the path of the charged particle, the secondary longitudinal electric-field $\mathcal{E}(\tau) \equiv E_z^{(s)}$ $(r=h,\ \phi=\phi_0,z,t)$ is given by

$$\mathcal{E}(\tau)=\frac{2q}{4\pi\epsilon_0 R^2}\sum_{\nu=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega(-j\Omega)\times\left[\frac{K_\nu\left(\frac{\Omega}{\gamma\beta}\frac{h}{R}\right)}{\dot{K}_\nu\left(\frac{\Omega}{\gamma\beta}\right)}\right]^2\frac{\mathcal{N}_\nu(\Omega)}{\mathcal{D}_\nu(\Omega)}\epsilon^{j\Omega(\tau c/R)}, \quad (11)$$

wherein $\tau \equiv t - z/V$ and

$$\mathcal{N}_\nu(\Omega)\equiv\left(\frac{\nu\gamma}{\Omega}\right)^2\left(1+\frac{1}{\epsilon\gamma^2\beta^2}\right)^2\times J_\nu^2(\Omega\sqrt{\epsilon})I_\nu(\Omega/\gamma\beta)K_\nu(\Omega/\gamma\beta)-\dot{I}_\nu(\Omega/\gamma\beta)\dot{K}_\nu(\Omega/\gamma\beta)\times\left[J_\nu(\Omega\sqrt{\epsilon})+\frac{\epsilon}{\gamma\beta\sqrt{\epsilon}}\dot{J}_\nu(\Omega\sqrt{\epsilon})\frac{I_\nu(\Omega/\gamma\beta)}{\dot{I}_\nu(\Omega/\gamma\beta)}\right]\times\left[J_\nu(\Omega\sqrt{\epsilon})+\frac{1}{\gamma\beta\sqrt{\epsilon}}\dot{J}_\nu(\Omega\sqrt{\epsilon})\frac{K_\nu(\Omega/\gamma\beta)}{\dot{K}_\nu(\Omega/\gamma\beta)}\right], \quad (12)$$

$$\mathcal{D}_\nu(\Omega)\equiv(\gamma\beta)^2\left\{\left(\frac{\nu\gamma}{\Omega}\right)^2\left(1+\frac{1}{\gamma^2\beta^2\epsilon}\right)^2 J_\nu^2(\Omega\sqrt{\epsilon})\times\left[\frac{K_\nu(\Omega/\gamma\beta)}{\dot{K}_\nu(\Omega/\gamma\beta)}\right]^2-\left[J_\nu(\Omega\sqrt{\epsilon})+\frac{\epsilon}{\gamma\beta\sqrt{\epsilon}}\times\dot{J}_\nu(\Omega\sqrt{\epsilon})\frac{K_\nu(\Omega/\gamma\beta)}{\dot{K}_\nu(\Omega/\gamma\beta)}\right]\times\left[J_\nu(\Omega\sqrt{\epsilon})+\frac{1}{\gamma\beta\sqrt{\epsilon}}\times\dot{J}_\nu(\Omega\sqrt{\epsilon})\frac{K_\nu(\Omega/\gamma\beta)}{\dot{K}_\nu(\Omega/\gamma\beta)}\right]\right\}. \quad (13)$$

We shall now investigate this wake field in the limiting case $\gamma \gg 1$; for this purpose, the expression occurring in Eq. (11) is divided into two parts: (a) that comprising the circular harmonic of order zero $(\nu=0)$; (b) all the remaining harmonics $(\nu\neq 0)$.

III. ANALYSIS OF THE SOLUTION

A. Circular harmonic of zero order $(\nu=0)$

The explicit expression for the wake for this case is

$$\mathcal{E}(\tau)=\frac{2q}{4\pi\epsilon_0}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega(-j\Omega)\left[\frac{K_0\left(\frac{\Omega}{\gamma\beta}\frac{h}{R}\right)}{\dot{K}_0\left(\frac{\Omega}{\gamma\beta}\right)}\right]^2\times\frac{\mathcal{N}_0(\Omega)}{\mathcal{D}_0(\Omega)}e^{j\Omega(c\tau/R)}, \quad (14)$$

where

$$\frac{\mathcal{N}_0}{\mathcal{D}_0}\simeq-\frac{1}{\gamma^2}I_1\left(\frac{\Omega}{\gamma}\right)K_1\left(\frac{\Omega}{\gamma}\right)\times\frac{J_0(\gamma\sqrt{\epsilon})-\frac{\epsilon}{\gamma\sqrt{\epsilon}}J_1(\Omega\sqrt{\epsilon})\frac{I_0(\Omega/\gamma)}{I_1(\Omega/\gamma)}}{J_0(\Omega\sqrt{\epsilon})+\frac{\epsilon}{\gamma\sqrt{\epsilon}}J_1(\Omega\sqrt{\epsilon})\frac{K_0(\Omega/\gamma)}{K_1(\Omega/\gamma)}}. \quad (15)$$

In order to evaluate the integral in Eq. (14), we shall determine the poles of the integrand at the limit $\gamma\to\infty$; these poles, in turn, are determined by the zeroes of the Bessel function $[p_s:\ J_0(p_s)\equiv 0,\ s=1,2,\ldots,\infty]$ hence,

$$J_0(\Omega\sqrt{\epsilon})=J_0(p_s)+(\Omega^2-\Omega_s^2)\left[\frac{d}{d\Omega^2}J_0(\Omega\sqrt{\epsilon})\right]_{\Omega=\Omega_s}, \quad (16)$$

with $\Omega_s \equiv p_s/\sqrt{\epsilon}$ and where $J_0(p_s)\equiv 0$. The main contribution to the integral in Eq. (14) is assumed to come from the poles associated with the denominator in Eq. (15) and therefore, Eq. (14) reads

$$\mathcal{E}(\tau)\simeq\frac{-q}{4\pi\epsilon_0 R^2}\frac{2}{\gamma^2}\frac{2\epsilon}{\epsilon-1}\sum_{s=1}^{\infty}\left[\frac{K_0\left(\frac{\Omega_s}{\gamma}\frac{h}{R}\right)}{K_1\left(\frac{\Omega_s}{\gamma}\right)}\right]^2\times\frac{d}{d\bar{\tau}}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega\frac{e^{j\Omega\bar{\tau}}}{\Omega^2-\Omega_s^2}\right\}, \quad (17)$$

where $\bar{\tau}\equiv\tau c/R$; in this expression we resort to the relation $I_0(u)K_1(u)+I_1(u)K_0(u)=1/u$. Imposing the requirements of causality, the last integral is recast into the analytic form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega\frac{e^{j\Omega\bar{\tau}}}{\Omega^2-\omega_s^2}=\frac{-1}{\Omega_s}\sin(\Omega_s\bar{\tau})\Theta(\bar{\tau}), \quad (18)$$

with $\Theta(u)$ denoting the Heaviside step function. Based upon this result, it is now possible to obtain the longitudinal wake field, $\mathcal{E}_0 \equiv \mathcal{E}(\overline{\tau}=0)$, explicitly given by

$$\mathcal{E}_0 = \frac{q}{4\pi\epsilon_0 R^2}\,\frac{4}{\gamma^2}\,\frac{\epsilon}{\epsilon-1}\sum_{s=1}^{\infty}\left[\frac{K_0\left(\frac{\Omega_s}{\gamma}\frac{h}{R}\right)}{K_1\left(\frac{\Omega_s}{\gamma}\right)}\right]^2. \tag{19}$$

A rough simplification of Eq. (19) may be obtained assuming a very large radius of curvature for the cylinder. Subject to this condition, i.e., $R \gg h-R$ and $h \gg h-R$, we obtain

$$\left[\frac{K_0\left(\frac{\Omega_s}{\gamma}\frac{h}{R}\right)}{K_1\left(\frac{\Omega_s}{\gamma}\right)}\right]^2 \simeq \frac{R}{h}e^{-(2\Omega_s/\gamma)[(h-R)/R]}, \tag{20}$$

and bearing in mind that for large arguments, the zero-order Bessel function of the first kind has *periodic* zeros, it is possible to approximate $\Omega_s = p_s/\sqrt{\epsilon-1} \sim \pi s/\sqrt{\epsilon-1}$, therefore obtaining

$$\sum_{s=1}^{\infty} e^{-s\{(2\pi/\gamma\sqrt{\epsilon-1})[(h-R)/R]\}} = \frac{1}{e^{(2\pi/\gamma\sqrt{\epsilon-1})[(h-R)/R]}-1} \simeq \frac{1}{\frac{2\pi}{\gamma\sqrt{\epsilon-1}}\frac{h-R}{R}}, \tag{21}$$

so that

$$\mathcal{E}_0 \sim \frac{\lambda}{4\pi\epsilon_0(h-R)}\,\frac{4\epsilon}{\sqrt{\epsilon-1}}\,\frac{1}{\gamma}, \tag{22}$$

where $\lambda = q/2\pi h$ denotes the charge-per-unit length. This expression clearly indicates that the decelerating force is inversely proportional to particle's momentum (recall that we assumed $\gamma \gg 1$). This runs contrary to the result occurring for the case of a point charge moving *inside* a symmetric tunnel bored in a dielectric material, in which case the decelerating force for $\gamma \gg 1$, is γ independent. However, this is almost exactly the expression for the decelerating field acting on a charged-line (λ) moving at a hight Δ from a dielectric half-space—this would correspond to a distance $h-R$ in the case investigated here. In Appendix A, it is shown that this field (denoted by $\mathcal{E}_\infty$) is given by

$$\mathcal{E}_\infty = \frac{\lambda}{4\pi\epsilon_0\Delta}\,\frac{2\gamma\beta\sqrt{\epsilon-\beta^{-2}}/\epsilon}{1+(\gamma\beta\sqrt{\epsilon-\beta^{-2}}/\epsilon)^2}, \tag{23}$$

which for a relativistic particle ($\gamma \gg 1$) simplifies to

$$\mathcal{E}_\infty \simeq \frac{\lambda}{4\pi\epsilon_0\Delta}\,\frac{2\epsilon}{\sqrt{\epsilon-1}}\,\frac{1}{\gamma}. \tag{24}$$

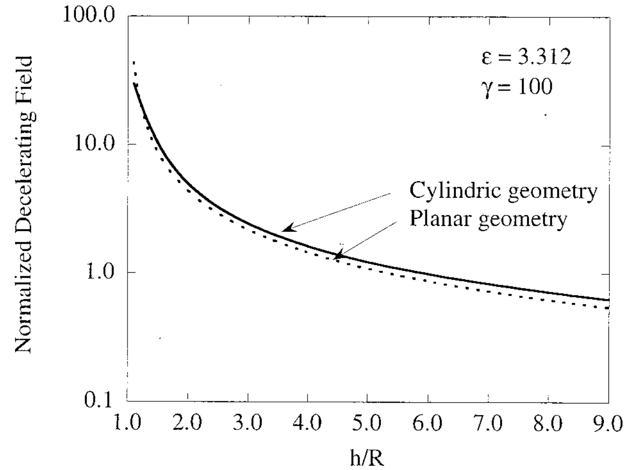

FIG. 2. The normalized decelerating field ($\nu=0$) as a function of the ratio h/R for $\gamma=100$ and $\epsilon=3.312$. Both the planar case ∞, expressed in Eq. (24) and the "exact" expression for the cylindrical configuration as given in Eq. (25), are shown to be very similar. The normalization in both cases is $(q/2\pi h)/4\pi\epsilon_0 h\gamma$. The number of Bessel harmonics that needs to be considered has to be of the order of $\gamma\sqrt{\epsilon-1}$, otherwise significant discrepancies occur. Subject to this condition, the similarity between the two curves is preserved even if the other two parameters are dramatically altered ($\gamma \sim 10\,800$ and $\epsilon=1.5,33$).

Up to a factor of 2, this exact expression is identical to the approximate result in Eq. (22). In fact, we found within a good approximation that

$$\mathcal{E}_0 = \frac{q/2\pi h}{4\pi\epsilon_0(h-R)}\left\{\frac{h}{R}\left(\frac{h}{R}-1\right)\frac{8\pi}{\gamma^2}\,\frac{\epsilon}{\epsilon-1} \times \sum_{s=1}^{\infty}\left[\frac{K_0\left(\frac{p_s}{\gamma\sqrt{\epsilon-1}}\frac{h}{R}\right)}{K_1\left(\frac{p_s}{\gamma\sqrt{\epsilon-1}}\right)}\right]^2\right\} \simeq \frac{\lambda}{4\pi\epsilon_0(h-R)}\left\{\frac{2\epsilon}{\sqrt{\epsilon-1}}\,\frac{1}{\gamma}\,\frac{h}{R}\right\}, \tag{25}$$

which is identical to Eq. (24) at the limit $h \gg h-R$. Figure 2 illustrates the exact decelerating field as represented by the first line in Eq. (25) normalized to $(q/2\pi h)/4\pi\epsilon_0 h\gamma$ as a function of the ratio h/R. As a reference, the planar case ($\mathcal{E}_\infty$) is also plotted and the similarity between the two is evident.

From the poles determined in Eq. (16), we may also deduce an important characteristic of the wake trailing the particle for active dielectric material, i.e., the dielectric coefficient consists of a real part ϵ_r, as well as an imaginary part ϵ_i that is nonzero within a limited frequency range ($\epsilon=\epsilon_r + j\epsilon_i$). Bearing in mind that $\Omega_s = p_s/\sqrt{\epsilon}$, the dimensionless growth rate in this case is given by

$$\frac{|\mathrm{Im}(\omega)|}{\omega_{\mathrm{res}}}=\sqrt{\frac{1}{2}}\sqrt{\frac{\epsilon_r-1}{(\epsilon_r-1)^2+\epsilon_i^2}}\sqrt{\sqrt{(\epsilon_r-1)^2+\epsilon_i^2}-\epsilon_r+1}$$
$$\sim\frac{\epsilon_i}{2(\epsilon_r-1)^{3/2}}, \tag{26}$$

where ω_{res} denotes the resonant circular frequency of the medium corresponding to one of the eigenfrequencies of the system. It is important to point out that this growth rate is γ-independent.

B. Nonzero circular harmonics ($\nu\neq 0$)

In this section, we shall consider the contribution of the circular harmonics of nonzero order. According to previous definitions and assuming $\gamma\gg 1$

$$\frac{\mathcal{N}_\nu}{\mathcal{D}_\nu}\simeq\nu\left(\frac{\nu\gamma}{\Omega}\right)\frac{K_\nu(\Omega/\gamma)}{\dot{K}_\nu(\Omega/\gamma)}\frac{1}{\Omega^2-[\nu\gamma K_\nu(\Omega/\gamma)/\dot{K}_\nu(\Omega/\gamma)]^2}, \tag{27}$$

here, it was tacitly assumed that the poles corresponding to circular modes are given by $\Omega^2=\nu^2\gamma^2K_\nu^2/\dot{K}_\nu^2\sim\nu^2\gamma^2$. Consequently, the longitudinal electric field acting back on the charge is given by the expression

$$\mathcal{E}=\frac{2q}{4\pi\epsilon_0R^2}\frac{d}{d\tau}\left\{\sum_{\nu\neq 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\Omega e^{j\Omega\tau}\left[\frac{K_\nu\left(\frac{|\Omega|}{\gamma}\frac{h}{R}\right)}{K_\nu\left(\frac{|\Omega|}{\gamma}\right)}\right]^2\right.$$
$$\left.\times\left[\frac{K_\nu\left(\frac{|\Omega|}{\gamma}\right)}{\dot{K}_\nu\left(\frac{|\Omega|}{\gamma}\right)}\right]^3\frac{\nu(\nu\gamma/|\Omega|)}{\Omega^2-\nu^2\gamma^2}\right\}. \tag{28}$$

When evaluated near the poles, the cubic term in the integrand is of order unity, therefore, following a similar approach as in the previous section, the decelerating electric field reads

$$\mathcal{E}=\frac{q}{4\pi\epsilon_0R^2}\left\{4\sum_{\nu=1}^{\infty}\left[\frac{K_\nu\left(\nu\frac{h}{R}\right)}{K_\nu(\nu)}\right]^2\right\}. \tag{29}$$

Using the explicit expression (see Ref. [14]) for asymptotic behavior for large orders, namely, $K_\nu(\nu\xi)\simeq\sqrt{\pi/2\nu}1/(1+\xi^2)^{1/4}e^{-\nu\eta(\xi)}$and $\eta(\xi)=\sqrt{1+\xi^2}+\ln[\xi/(1+\sqrt{1+\xi^2})]$ it is shown in Appendix B that for $R\gg h-R$, the decelerating field may be approximated by the relatively simple equation

$$\mathcal{E}\simeq\frac{q}{4\pi\epsilon_0R^2}\frac{1}{2\left(\frac{h}{R}-1\right)^2} \tag{30}$$

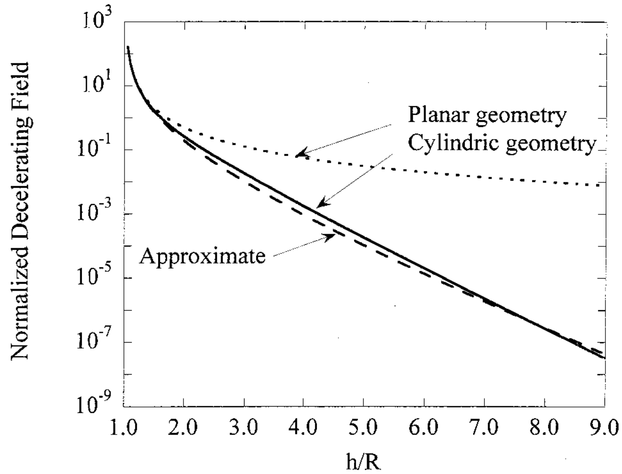

FIG. 3. The normalized decelerating field ($\nu\neq 0$) as a function of the ratio h/R. For $\gamma\gg 1$, this field is γ independent. Both the planar case ∞, expressed in Eq. (31) and the ''exact'' expression for the cylindrical configuration as given in Eq. (29) are illustrated here. Both curves combine into one for $h/R\to 1$ but they strongly diverge for $h/R>3$. An approximated expression for the decelerating field is suggested in Eq. (32). As clearly revealed by the dashed line, it fits well the ''exact'' expression. All quantities are normalized to $q/4\pi\epsilon_0R^2$.

This expression is identical with the reaction field [see Ref. [15]] on a point charge moving at a height $\Delta(=h-R)$ above a dielectric half space when $\gamma\gg 1$, which explicitly reads

$$\mathcal{E}_\infty\simeq\frac{q}{4\pi\epsilon_0(2\Delta)^2}\times 2. \tag{31}$$

Figure 3 illustrates the asymptotic expression ($\mathcal{E}_\infty$) and the expression in Eq. (29)—both normalized to $q/4\pi\epsilon_0R^2$. It is evident that as the ratio h/R is closer to unity, the decelerating field is inversely proportional to the square of the height from the surface of the cylinder. Contrary to the result in the previous section, there is great discrepancy between the asymptotic solution ($\mathcal{E}_\infty$) and the exact one for $h/R>3$. Based on these two expressions, we developed an approximation of the exact solution that has the following form:

$$\mathcal{E}_{\mathrm{approx}}\simeq\frac{q}{4\pi\epsilon_0R^2}\frac{1}{2\left(\frac{h}{R}-1\right)^2\cosh\left[\frac{3}{2}\left(\frac{h}{R}-1\right)\right]}, \tag{32}$$

this expression is also illustrated in Fig. 3 and evidently, it shows excellent agreement with the exact solution.

The expression in Eqs. (29) and (32) clearly imply a γ-independent decelerating field for very high γ. Moreover, the high-order ($\nu>0$) wake behind the particle is independent of the dielectric coefficient, implying that in case of an *active* dielectric, the wake does not grow behind the particle.

Consequently, in zero order, it does not contribute to the possible acceleration of a trailing bunch. Another interesting feature revealed by the current analysis is associated with the eigenfrequency of the high-order modes $-\Omega=\pm\nu\gamma$. With such eigenfrequencies, the mode is slowly gyrating. Bearing in mind that the phase of the wave is determined by $\exp[\pm j\nu\gamma(t-z/V)c/R-j\nu(\phi-\phi_0)]$ we observe that there are two families of high-order modes gyrating clockwise or counter-clockwise, each family having its own characteristic phase velocity.

C. Finite size bunch $(\nu=0)$

Having determined the wake generated by a point charge, we proceed one step further and determine the wake of a finite-size bunch. Since we are interested in the potential of this wake to accelerate a particle in the presence of the active dielectric, we shall consider only the contribution of the zero harmonic. Specifically, we assume a bunch of length Δ_z and radial width Δ_r; the wake behind the particle is given therefore by

$$\mathcal{E}(\tau,r)=\sum_{s=1}^{\infty}\mathcal{E}_s K_0\left(\frac{\Omega_s}{\gamma}\frac{r}{R}\right)\left\{\frac{2}{h_+^2-h_-^2}\int_{h_-}^{h_+}dr'r'K_0\left(\frac{\Omega_s}{\gamma}\frac{r'}{R}\right)\right\}\times\left\{\frac{c}{\Delta_z}\int_{\tau_-}^{\tau_+}d\tau'\cos(\Omega_s\tau'c/R)\Theta(\tau')\right\}, \tag{33}$$

where it was tacitly assumed that $v\simeq c$, $h_\pm\equiv h\pm\Delta_r/2$, $\tau_\pm\equiv\tau\pm\Delta_z/2c$, and

$$\mathcal{E}_s\equiv\frac{q}{4\pi\epsilon_0R^2}\frac{4}{\gamma^2}\frac{\epsilon}{\epsilon-1}\frac{1}{K_1^2(\Omega_s/\gamma)}. \tag{34}$$

Both integrals in the curled brackets may be evaluated analytically

$$\mathcal{H}_s\equiv\frac{2}{h_+^2-h_-^2}\int_{h_-}^{h_+}dr'r'K_0\left(\frac{\Omega_s}{\gamma}\frac{r'}{R}\right)=\frac{2}{\xi_+^2-\xi_-^2}[\xi_-K_1(\xi_-)-\xi_-K_1(\xi_+)], \tag{35}$$

where $\xi_\pm\equiv(\Omega_s/\gamma)(h_\pm/R)$ and

$$T_s(\tau)\equiv\frac{c}{\Delta_z}\int_{\tau_-}^{\tau_+}d\tau'\cos(\Omega_s\tau'c/R)\Theta(\tau')=\begin{cases}0 & \text{for } \tau<-\Delta_x/2V,\\ \sin(\Omega_s\tau_+c/R)/(\Omega_s\Delta_z/R) & \text{for } |\tau|<\Delta_z/2V,\\ \sin(\Omega_s\tau_+c/R) & \\ -\sin(\Omega_s\tau_-c/R)/(\Omega_s\Delta_z/R) & \text{for } \tau>\Delta_z/2V.\end{cases} \tag{36}$$

Based on these explicit expressions, we are able to determine the expression of the electro-magnetic power generated by the bunch, since $P=\int dv\, J_zE_z$ and therefore

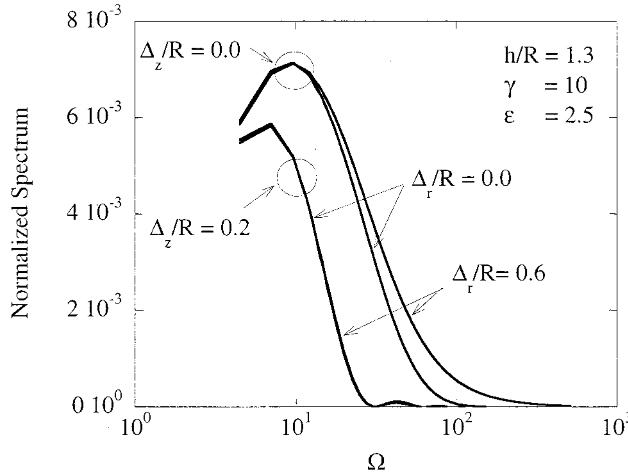

FIG. 4. The normalized spectrum as a function of the frequency for $h/R=1.3$, $\gamma=10$, $\epsilon=2.5$. The longitudinal and radial extensions of the bunch are the two parameters of this plot.

$$\overline{P}\equiv\frac{P}{-q^2c/4\pi\epsilon_0R^2}=\frac{2}{\gamma^2}\frac{\epsilon}{\epsilon-1}\sum_{s=1}^{\infty}\frac{\mathcal{H}_s^2}{K_1^2(\Omega_s/\gamma)}\operatorname{sinc}^2\left(\frac{1}{2}\Omega_s\frac{\Delta_z}{R}\right), \tag{37}$$

wherein $\operatorname{sinc}(x)\equiv\sin(x)/x$.

Figure 4 illustrates the normalized spectrum of the emitted power, i.e.,

$$\overline{P}_s=\frac{2}{\gamma^2}\frac{\epsilon}{\epsilon-1}\frac{\mathcal{H}_s^2}{K_1^2(\Omega_s/\gamma)}\operatorname{sinc}^2\left(\frac{1}{2}\Omega_s\frac{\Delta_z}{R}\right). \tag{38}$$

It may be seen that for a relativistic bunch (e.g., $\gamma=10$) the location of the peak of the spectrum is virtually not dependent on the width (Δ_r) of the bunch; however, it is strongly dependent on its length (Δ_z). For a very short bunch $(\Delta_z=0)$ the spectrum is wider in case the radial width of the bunch is larger. This may readily be understood, since the bunch consists of electrons closer to the dielectric cylinder that widen the spectrum. Note also that when the bunch *length* is a significant fraction of R, the radial width of the bunch virtually does not affect the spectrum of the emitted radiation; the lower curve $(\Delta_z=0)$ represents actually two indistinguishable curves ($\Delta_r=0$ and $0.6R$). This result is even better illustrated in Fig. 5, which indicates that for $\Delta_z\geqslant0.2R$, the normalized power $[\overline{P}_n\equiv\overline{P}(\Delta_z,\Delta_r)/\overline{P}(\Delta_z=0,\Delta_r=0)]$ is virtually not dependent on the radial width of the bunch.

IV. SUMMARY

The decelerating field linked with the Cerenkov radiation generated by a relativistic $(\gamma\gg1)$ bunch of electrons moving parallel to a dielectric cylinder, was analyzed. The overall effect of the emitted electromagnetic energy on the moving particle was split into two contributions: the zero-order circular harmonic, which contributes a decelerating force in-

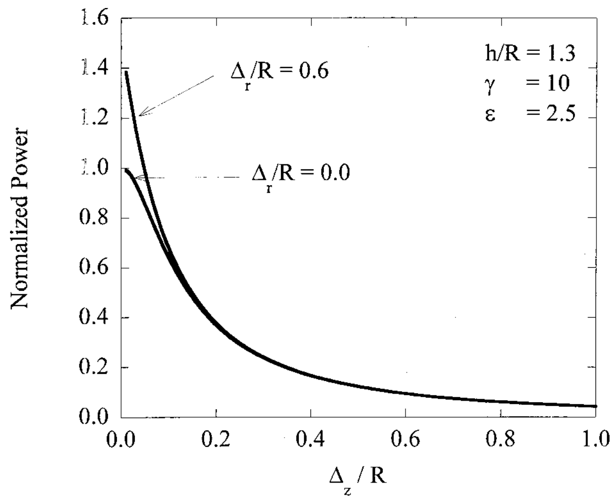

FIG. 5. The normalized power $[\overline{P}_n \equiv \overline{P}(\Delta_z, \Delta_r)/\overline{P}(\Delta_z = 0, \Delta_r = 0)]$ is virtually independent of the radial width of the bunch for values $\Delta_z \geqslant 0.2R$.

versely proportional to γ, and the nonzero circular harmonics contributing a γ-independent force—Eq. (32) is a good approximation for all practical purposes provided $\gamma \gg 1$. Moreover, the wake linked to the zero-order circular harmonic may grow in space if the dielectric cylinder consists of an active medium; the growth rate is γ independent. For highly relativistic particles, the wake attached to the nonzero circular harmonics does not grow in space in case the medium is active. For a bunch not smaller than any typical dimension of the system, the normalized emitted power decreases algebraically with the length of the bunch, and not exponentially. In addition, the width of the bunch causes an increase in the width of the spectrum of the generated waves.

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APPENDIX A

Consider a charged-line (λ) moving at a height Δ above a dielectric (ϵ) half space. The moving charge generates a current density in the z direction given by

$$J_z(x,z,t) = -\lambda V \delta(x-\Delta)\,\delta(z-Vt). \tag{A1}$$

In the absence of the dielectric medium, this current density generates an electromagnetic field that may be derived from the following (primary-superscript p) magnetic vector potential

$$A_z^{(V)}(x,z,t) = -\frac{\mu_0 \lambda}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{1}{2\Gamma}\, e^{-\Gamma|x-\Delta|} e^{j\omega(t-z/V)}, \tag{A2}$$

wherein $\Gamma = |\omega|/\gamma\beta c$.

The presence of the dielectric half space in the region $x < 0$ changes the electromagnetic-field distribution. In the upper-half space (vacuum), this is given by

$$A_z^{(s)}(x>0,z,t) = -\frac{\mu_0 \lambda}{2\pi}\int_{-\infty}^{\infty} d\omega\, A(\omega)\, \frac{1}{2\Gamma}\, e^{-\Gamma x} e^{j\omega(t-z/V)}, \tag{A3}$$

whereas in the lower-half space, the dielectric material imposes a solution of the form

$$A_z^{(s)}(x<0,z,t) = -\frac{\mu_0 \lambda}{2\pi}\int_{-\infty}^{\infty} d\omega\, B(\omega)\, \frac{1}{2\Gamma}\, e^{j\Lambda x} e^{j\omega(t-z/V)}; \tag{A4}$$

here, $\Lambda = |\omega|\sqrt{\epsilon - \beta^{-2}}/c$; the superscript ($s$) emphasizes that this is a secondary field. In order to determine the two amplitudes $A(\omega)$ and $B(\omega)$, the continuity of the tangential field is imposed at $x=0$. Explicitly,

$$A(\omega) = -\frac{1 - j\sqrt{\epsilon-\beta^{-2}}\,\gamma\beta/\epsilon}{1 + j\sqrt{\epsilon-\beta^{-2}}\,\gamma\beta/\epsilon}\, e^{-\Gamma\Delta} \tag{A5}$$

is the expression for the amplitude in the upper-half space that allows us to determine the decelerating field

$$\begin{aligned}\mathcal{E}_\infty &\equiv E_s^{(s)}\quad (x=h,\ z=Vt,t)\\ &= -\frac{\mu_0\lambda}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{c^2}{j\omega}\left(\frac{\omega^2}{c^2} - \frac{\omega^2}{v^2}\right) A(\omega)\, \frac{1}{2\Gamma}\, e^{-\Gamma\Delta}.\end{aligned} \tag{A6}$$

Finally, the decelerating field corresponding to the Cerenkov condition ($\epsilon > \beta^{-2}$) is given by

$$\mathcal{E}_\infty = \frac{\lambda}{4\pi\epsilon_0 \Delta}\, \frac{2\sqrt{\epsilon-\beta^{-2}}\,\gamma\beta/\epsilon}{1 + (\sqrt{\epsilon-\beta^{-2}}\,\gamma\beta/\epsilon)^2}, \tag{A7}$$

otherwise ($\epsilon < \beta^{-2}$), this field is zero.

APPENDIX B

Our starting point is Eq. (29) and our purpose is to develop an expression for the asymptotic behavior of the decelerating field for a cylinder of large curvature ($R \gg h - R$). For this purpose, we use an asymptotic expression for large orders of the modified Bessel functions of the second kind of order ν, namely,

$$K_\nu(\nu\xi) \simeq \sqrt{\frac{\pi}{2\nu}}\, \frac{1}{(1+\xi^2)^{1/4}}\, e^{-\nu\eta(\xi)}, \tag{B1}$$

wherein, $\eta(\xi) = \sqrt{1+\xi^2} + \ln[\xi/(1+\sqrt{1+\xi^2})]$. Substituting in Eq. (29), we obtain

$$\mathcal{E} = \frac{q}{4\pi e_0 R^2}\left\{4\sum_{\nu=1}^{\infty} \nu\left[\frac{2^{1/4}}{[1+(h/R)^2]^{1/4}}\, e^{-\nu[\eta(h/R)-\eta(1)]}\right]^2\right\}. \tag{B2}$$

The sum may be readily evaluated as

$$\mathcal{E}=\frac{q}{4\pi\epsilon_0 R^2}\times 4\sqrt{\frac{2}{1+(h/R)^2}}\frac{e^{2[\eta(h/R)-\eta(1)]}}{(e^{2[\eta(h/R)-\eta(1)]}-1)^2}, \tag{B3}$$

therefore expanding for small values above unity, i.e., $h/R=1+\delta$ where $\delta\ll 1$, we obtain

$$\eta(h/R)-\eta(1)=\sqrt{2}\left(\frac{h}{R}-1\right) \tag{B4}$$

and finally,

$$\mathcal{E}=\frac{q}{4\pi\epsilon_0 R^2}\frac{1}{2\left(\frac{h}{R}-1\right)^2}. \tag{B5}$$

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